MODIFIED NEWTON–RAPHSON METHOD FOR SOLVING A SYSTEM OF NONLINEAR EQUATIONS IN PROBLEMS OF COMPLICATED HEAT EXCHANGE

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A method is proposed for solving the nonlinear system of equations arising from an algebraic approximation of equations of radiation and convection energy transport.

Prediction of the thermal regimes of high-temperature technical objects is usually based on a zonal method, which allows us to reduce the problem of simultaneous energy transfer by radiation and convection to the solution of the system of nonlinear algebraic equations [1]:

$$\sum_{j=1}^{N} R_{ij}(T_j)^4 + \sum_{j=1}^{N} A_{ij}T_j + S_i = 0,\ i = \overline{1,\ N}. \tag{1}$$

The matrix of the coefficients of convective exchange A_{ij} is asymmetrical, it has a predominance of diagonal elements and a disperse (sparse) structure: The number of nonzero elements in the i-th row is determined by the number of the neighbors that interact with the i-th zone. The matrix of the coefficients of radiative exchange R_{ij} in the general case is asymmetrical, it has diagonal predominance and is filled completely (with nonzero elements). However, for most practical cases this matrix has a block structure: Each block with a dimensionality of $N_b \times N_b$ corresponds to a subsystem from N_b zones, closed with respect to radiation, which interacts with other subsystems only by means of convection and heat conduction. In addition, if we can neglect the selectivity of radiation of the medium and the boundaries, then the matrix R_{ij} is symmetrical.

Of a host of possible methods [1-3] of the solution of the system of Eqs. (1) the Newton–Raphson method is most often used. In its classical formulation this method assumes an exact solution of the system of linearized equations on each iteration. Apparently, this approach is the best when the number of equations in system (1) does not exceed 100. However, even here one can encounter difficulties, if the initial point lies too far from the solution. As for high-dimension problems, the exact solution of a linearized system by the Gauss method of elimination is onerous from the point of view of time and memory. Because of the inevitable linearization errors it is not necessary to obtain an exact solution for each iteration, and for obtaining an approximate solution it is better to use iteration methods, for example, the method of conjugate gradients [5, 6].

In the present work, a modified version of the Newton–Raphson method is used, which is free from the drawbacks mentioned above. The new method provides for considerable savings in calculations due to the use of the method of conjugate gradients for solving the linearized system of equations.

In the classical Newton–Raphson method, at first an approximate value for each temperature T_i^0 is specified and the discrepancies are determined:

$$\xi_i^0 = \sum_{j=1}^{N} R_{ij}(T_i^0)^4 + \sum_{j=1}^{N} A_{ij}T_j^0 + S_i. \tag{2}$$

Then the corrections ΔT_i^0 are calculated so that $T_i^1 = T_i^0 + \Delta T_i^0$. The values of ΔT_i^0 are determined by solving the following system of equations, which is the result of linearization of (1):

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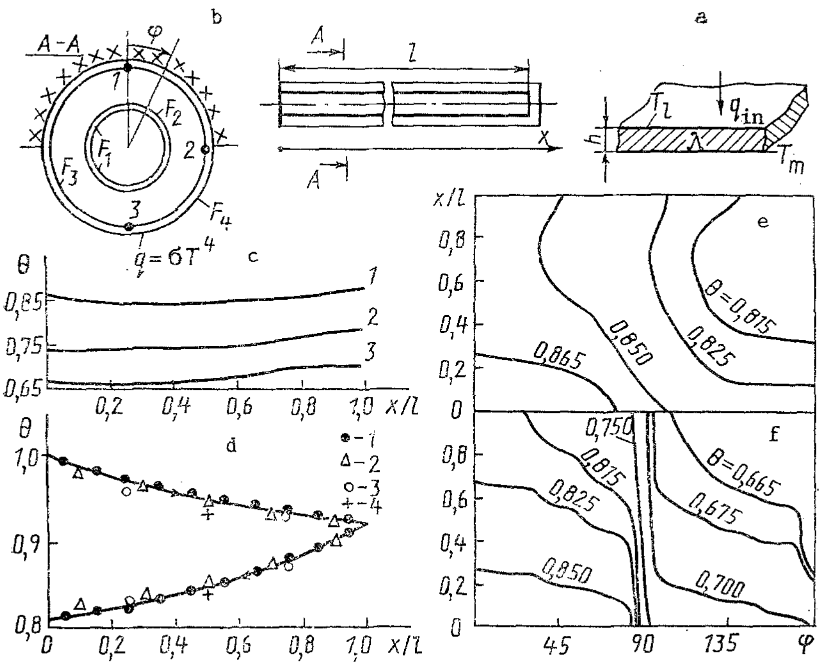

Fig. 1. Conditional diagram and distribution of relative temperatures in a sample: a) model of a thin screen; b) diagram of a radiation pipe; c) temperature distribution on the inner surface of the external pipe at three points shown in Fig. 1a; d) variation of the relative temperature of the heat carrier along the direction of its motion: (1) calculated values of the relative temperature for the pipe divided into ten regions, (2) calculated temperatures for the pipe divided into five regions, (3) calculated temperatures for the pipe divided into two regions, (4) calculated temperatures for the pipe considered as one region; e) isotherms of the surface F_2; f) isotherms of the surface F_3, φ, deg.

$$\sum_{j=1}^{N} [4R_{ij}(T_j^0)^3 + A_{ij}]\,\Delta T_j^0 = -\xi_i^0, \quad i = \overline{1, N}, \tag{3}$$

or in the vector form

$$\|B\|\,\Delta \mathbf{T}^0 = -\xi^0. \tag{3'}$$

The values of T_i^1, determined in this way, are used for the calculation of a new correction and the process is continued until the condition of convergence is attained: $(\sum_{i=1}^{N}(\xi_i^0)^2/N)^{1/2} \le \gamma$.

New Variables. As usual, the system of Eqs. (3) is ill-conditioned and, therefore, it is of little use for solving by iteration methods. We carry out the substitution of variables:

$$\Delta X_i^0 = [4R_{ii}(T_i^0)^3 + A_{ii}]\,\Delta T_i^0. \tag{4}$$

By substituting the values of ΔT_i from (4) in (3), we arrive at a system of equations linear with respect to ΔX_i with a preconditioned matrix. Such a system is well-suited for solving by the method of conjugate gradients by using the standard scheme with symmetrization [6]. As a result, we find the unknown values of increments ΔX_i^0. If we now use Eq. (4) for the determination of the corrections ΔT_i^0, then we return to the classic Newton–Raphson scheme, which diverges for a poor choice of T_i^0. The stability factor of the method with respect to the choice of the initial approximation can be improved slightly, if we consider (4) as a linearization of the following relationship, which holds exactly:

$$X_i = R_{ii}T_i^4 + A_{ii}T_i. \tag{5}$$

Therefore, by knowing the increments ΔX_i^0, we determine the values of $X_i^1 = X_i^0 + \Delta X_i^0$. After this, for the determination of new values of the temperature T_i^1, we solve the nonlinear equations

$$R_{ii}(T_i^1)^4 + A_{ii}T_i^1 = X_i^1, \; i = \overline{1, N}. \tag{6}$$

Each of these equations is solved separately with the help of the Newton method. The temperature values that are determined in this way are used for the calculation of a new discrepancy (3) and the process is continued until convergence is attained.

Physical Systems with Screens. The modified Newton–Raphson method that we have considered is very effective if the system of Eqs. (2) is not too "rigorous." There are a number of practically important problems, which can be solved by the aforementioned method. The problem of complicated heat exchange in a physical system with a screen, which has a low thermal resistance, can serve as an example.

Before we consider the general algorithm, we explain the essence of the difficulties that arise when solving the system of Eqs. (1). In Fig. 1a we show a plate that is heated at the top by the flow of incident radiation q_{in}. Both sides of the plate are absolutely black and radiate freely in the surrounding medium. The system of equations of heat balance for the upper and lower sides of the plate is of the form (for 1 m^2 of the surface)

$$\begin{aligned} &-\sigma T_l^4 - D_\lambda T_l + D_\lambda T_m + q_{in} = 0,\\ &-\sigma T_m^4 + D_\lambda T_l - D_\lambda T_m = 0, \end{aligned} \tag{7}$$

where $D_\lambda = \lambda/h$. If $D_\lambda \to \infty$, then $T_l \to T_m$, and the linearized system (7) is ill-conditioned even after the substitution of variables (4). In connection with this, we consider another method of transformation of the system matrix. If we add the first of Eqs. (7) to the second equation and subtract the first equation from the second equation, we convert to the equivalent system

$$\begin{aligned} &\sigma(T_l^4 + T_m^4) + q_{in} = 0,\\ &\sigma(T_l^4 - T_m^4) - D_\lambda(T_l - T_m) - q_{in} = 0, \end{aligned} \tag{8}$$

which, after linearization and substitution of the variables (4) and (5), is well-conditioned and appropriate for solving by the method of conjugate gradients.

In a more general case, we consider a physical system with a screen, on the sides of which the surface zones l and m are situated. We isolate from the general system of Eqs. (1) the equation of heat balance for these two zones:

$$\sum_{j \neq l} R_{lj}T_j^4 + R_{ll}T_l^4 + A_{ll}T_l + \sum_{j \neq l} A_{lj}T_j + S_l = 0; \tag{9a}$$

$$\sum_{j \neq m} R_{mj}T_j^4 + R_{mm}T_m^4 + A_{mm}T_m + \sum_{j \neq m} A_{mj}T_j + S_m = 0 \tag{9b}$$

and replace them by the equivalent pair of equations

$$\begin{aligned} &\sum_{j \neq l,m} [(R_{lj} + R_{mj})T_j^4 + (A_{lj} + A_{mj})T_j] + (R_{ll} + T_{ml})T_l^4 +\\ &+ (R_{mm} + R_{lm})T_m^4 + (A_{ll} + A_{ml})T_l + (A_{mm} + A_{lm})T_m + S_l + S_m = 0; \end{aligned} \tag{10a}$$

(10a)

$$\begin{aligned} &\sum_{j \neq l,m} [(-R_{lj} + R_{mj})T_j^4 + (-A_{lj} + A_{mj})T_j] + (-R_{ll} + R_{ml})T_l^4 +\\ &+ (R_{mm} - R_{lm})T_m^4 + (-A_{ll} + A_{ml})T_l + (A_{mm} - A_{lm})T_m + S_m - S_l = 0. \end{aligned} \tag{10b}$$

Next the system of Eqs. (1), modified in this way, is linearized by Newton's method. The increments ΔT_i are replaced by the increments ΔX_i, which for $i \neq l$ are determined from (4). For $i = l, m$, substitution of the variables looks different:

$$\begin{aligned} X_l &= (R_{ll} + R_{ml}) T_l^4 + (A_{ll} + A_{ml}) T_l + \\ &+ (R_{lm} + R_{mm}) T_m^4 + (A_{lm} + A_{mm}) T_m; \\ X_m &= (-R_{lm} + R_{mm}) T_m^4 + (-A_{lm} + A_{mm}) T_m + \\ &+ (-R_{ll} + R_{ml}) T_l^4 + (-A_{ll} + A_{ml}) T_l. \end{aligned} \tag{11}$$

From this, we obtain the relationship between the increments ΔX_i and ΔT_i for $i = l, m$:

$$\begin{aligned} \Delta X_l &= e_{ll}\Delta T_l + e_{lm}\Delta T_m; \\ \Delta X_m &= e_{ml}\Delta T_l + e_{mm}\Delta T_m; \end{aligned} \tag{12}$$

$$\begin{aligned} \Delta T_l &= c_{ll}\Delta X_l + c_{lm}\Delta X_m; \\ \Delta T_m &= c_{ml}\Delta X_l + c_{mm}\Delta X_m, \end{aligned} \tag{13}$$

where

$$\begin{aligned} e_{li} &= 4(R_{li} + R_{mi})(T_i^0)^3 + A_{li} + A_{mi}; \\ e_{mi} &= 4(-R_{li} + R_{mi})(T_i^0)^3 - A_{li} + A_{mi}, \quad i = l,\ m; \\ c_{ll} &= e_{mm}/\Delta_{lm};\ c_{mm} = e_{ll}/\Delta_{lm}; \\ c_{lm} &= -e_{lm}/\Delta_{lm};\ c_{ml} = -e_{ml}/\Delta_{lm}; \\ \Delta_{lm} &= e_{ll}e_{mm} - e_{lm}e_{ml}. \end{aligned} \tag{14}$$

After we have substituted ΔX_i^0 for ΔT_i^0 from (4) or (13), we convert to a well-conditioned linear system, and, as a result of solving it, we determine ΔX_i^0 and the new values of $X_i^1 = X_i^0 + \Delta X_i^0$. For $i \neq l, m$, the new values of the temperature T_i^1 are determined after the nonlinear Eq. (6) has been solved, and for the determination of T_l^1 and T_m^1, the system of two equations (11) is solved by Newton's method.

The numerical experiments show that the change of variables (11)-(13) yields better results in the sense of convergence, as compared with (4) and (5), when the following conditions are simultaneously fulfilled:

$$\begin{cases} B_{lm}/|B_{mm}| \geq \delta, \\ B_{ml}/|B_{ll}| \geq \delta, \end{cases} \quad \delta = 0{,}8, \tag{15}$$

where B_{ij} are the coefficients of the linearized system of Eqs. (3').

Algorithm of the Method. We formulate next an algorithm for the proposed method, whose starting point is the linearized system of Eqs. (3'). An analysis of the coefficients of this system allows us to isolate pairs of the strongly interconnected zones, for which condition (15) is fulfilled. We call the two pairs of zones (l_1, m_1) and (l_2, m_2) nonintersecting, if the sequence of the zone numbers l_1, m_1, l_2, m_2 does not contain the same numbers. We denote the set of these pairs by $\Pi_2 = \{(l_1, m_1), (l_2, m_2), \ldots, (l_k, m_k)\}$, and the corresponding set of the zone numbers by $\Pi_1 = \{l_1, m_1, l_2, m_2, \ldots, l_k, m_k\}$. We define also the set Π_0 of the zone numbers that do not form pairs.

Preconditioning of the matrix of the linear system (3') is reduced to the substitution of the increments ΔX_i for the corrections ΔT_i. If $i \in \Pi_0$, then for the substitution of variables, we use (4); otherwise (13) is used. As a result, system (3') assumes the form

$$\|H\|\,\|B\|\,\|P\|\,\Delta \mathbf{X}^0 = -\xi^0, \tag{16}$$

where

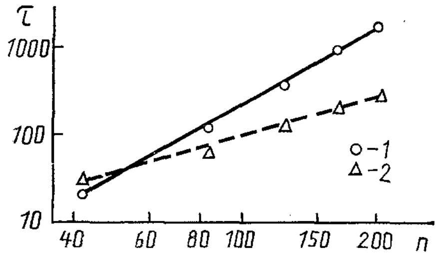

Fig. 2. Dependence of the calculation time for solving a test example on an ES-1033 computer on the number of zones: 1) Newton–Raphson method; 2) modified Newton–Raphson method.

$$H_{ij}=\begin{cases}1, & i=j;\\ 1, & (i,\ j)\in\Pi_2;\\ -1, & (j,\ i)\in\Pi_2;\\ 0, & i\neq j,\ i\in\Pi_0,\ j\in\Pi_0;\end{cases}\quad P_{ij}=\begin{cases}c_{ij}, & (i,\ j)\in\Pi_2;\\ c_{ji}(j,\ i)\in\Pi_2;\\ c_{ii}, & i=j,\ i\in\Pi_1;\\ (4R_{ii}(T_i^0)^3+A_{ii})^{-1}, & i=j,\ i\in\Pi_0;\\ 0, & i\neq j,\ i\in\Pi_0,\ j\in\Pi_0,\end{cases}$$

and the values of c_{ij} are taken from (14).

After symmetrization, the system of linear equations (16) is solved by the method of conjugate gradients. The new values of $X_i^1 = X_i^0 + \Delta X_i^0$ are used for the calculation of T_i^1 by means of solving separate nonlinear equations (6) (for $i \in \Pi_0$) or the system of nonlinear Eqs. (11), when $i \in \Pi_1$.

Introduction of a Barrier. The coefficients of radiation exchange R_{ij} are distributed along the column unevenly. Often the difference between the nondiagonal elements in the column constitute a few orders, which means that some zones are almost completely screened from the radiation of other zones. The cause for this might be the geometrical peculiarities of the object or the high optical density of the medium. If in the linearization of the system of Eqs. (1) we do not take account of weak interactions of the zones, then we can achieve a certain economy in calculations. With this in mind we introduce a relative barrier $\beta < 1$ and when forming the coefficients of the linear system (3), we discard the coefficients $R_{ij} < \beta|R_{jj}|$. The value of the relative barrier β does not depend on the particular problem.

Numerical Experiments. The problem selected for testing the method is similar in many ways to those encountered in the analysis of heat exchange in the radiation pipes or recuperators of industrial furnaces. In Fig. 1b, a heat-exchanging element is shown, which consists of an external dead-end pipe with a diameter of 2d and an internal pipe, open freely, with a diameter of d. A gas with initial temperature T'_g and constant absorption coefficient α enters the inner pipe and is removed through the ring gap between the pipes.

The pipes are made of a material with constant coefficient of thermal conductivity λ. The surfaces of heat exchange F_1, F_2, F_3, and F_4 (Fig. 1b) are absolutely black. The coefficient of heat emission from the gas to the surfaces of pipes is not considered. The lower half of the surface F_4 freely radiates in the open space, the upper half of this surface is adiabatic.

It is assumed that the pipe lengths are $l \gg d$, and the wall thicknesses are $h \ll D$, so that longitudinal heat transfer owing to radiation and heat conduction can be neglected. In addition, it is assumed that the gas is well-mixed and its temperature depends only on the coordinate x. For these conditions it is required to determine the gas temperature $T_g(x)$ and the temperatures of heat-exchanging surfaces $T_{si}(x, \varphi)$, i = 1, ..., 4. The solution of this problem can be represented in the dimensionless form $\Theta_g(x/l) = T_g/T'_g$; $\Theta_{si}(x/l, \varphi) = T_{si}/T'_g$ and depends on three dimensionless parameters:

$$\chi_1 = \alpha d;\ \chi_2 = W/(ld\sigma(T'_g)^3);\quad \chi_3 = \lambda/(h\sigma(T'_g)^3). \tag{17}$$

At present, the only method that allows us to solve problems of this type is a zonal method. A heat-exchanging element is divided lengthwise into the regions to be calculated, each of which is considered as a system, closed with respect to radiation. The accuracy

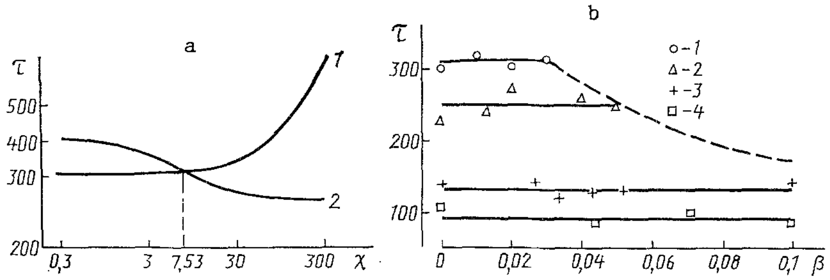

Fig. 3. Dependence of the calculation time for solving a test example on the parameters of the algorithm: a) dependence on the degree of the "coupling" of the zones ≪χ_3≫: [(1) using substitution of variables (4); 2) using substitution of variables (11)]; b) dependence on the value of the relative barrier β [(1), $\delta\theta = 5\cdot10^{-4}$; 2) 10^{-3}; 3) $5\cdot10^{-3}$; 4) 10^{-2}].

of the calculation depends both on the total number of calculated regions and on the degree of detailing of the discrete model for each of them.

In the given case, for the calculation of the coefficients R_{ij} in the system of Eqs. (1), the surfaces F_1 and F_2 were divided into 16 equal bands and the surface F_3, into 32 bands. The upper part of the surface F_4 was not considered since it did not participate in heat exchange. The lower part of this surface was divided into 16 bands. With consideration for symmetry, these bands were paired into surface zones. As a result, within each calculated region we isolated 40 surface zones. The total number of zones with consideration for two spatial zones was equal to 42.

Therefore, the complete model of the heat-exchanging element consisted of $N = 42\cdot N_x$ zones, where N_x is the number of the calculated regions. The unknown zone temperatures were determined by solving the system of nonlinear Eqs. (1). The results of the solution of the problem for $\chi_1 = 0.3$, $\chi_2 = 3$, $\chi_3 = 30$ and different $N_x = 1, 2, 5, 10$ are represented in Fig. 1c and d. The isotherms for the surface F_2 (Fig. 1e) and F_3 (Fig. 1f) were plotted on the basis of numerical simulation for $\chi_1 = 0.3$, $\chi_2 = 3$, $\chi_3 = 30$, $N_x = 10$.

The parameter χ_3 was taken large enough to make the temperature gradient on the walls of the pipes negligible. Satisfactory convergence of results to the solution of the problem is attained only for $N_x = 5$, and the total number of the unknowns in the system of Eqs. (1) is $N = 42 \times 5 = 210$.

In Fig. 2, the calculation time for the ES-1033 processor is shown for solving nonlinear Eqs. (1) as a function of the number of unknowns. The method proposed in the given work was compared to the classic Newton–Raphson method. Since both methods are iterative, at first the solution of the system of Eqs. (1) was carried out until complete convergence was attained, and the time required to attain an accuracy of $\delta\theta = 5\cdot10^{-4}$ was determined, where $\delta\theta$ is the maximal error in the determination of the dimensionless temperature. As is seen from Fig. 2, the calculation time according to the Newton–Raphson method for N = 210 is 7.5 times higher as compared with the method proposed in this work.

An investigation of the effect of preconditioning (4), (5), and (11)-(13) on the algorithm convergence for different values of the parameter χ_3 was carried out for the case (Fig. 1b) with the number of independent unknowns equal to 210 for $\chi_1 = 0.3$ and $\chi_2 = 3$. The results of simulation are shown in Fig. 3a. The intersection point of curves 1 and 2 corresponds to the value of δ from (15) equal to 0.792. In the given example the maximal temperature gradient along the thickness of the wall of the external pipe for $\chi_3 = 3$ is 363 K, for $\chi_3 = 300$-6 K, for $T'_g = 2000$ K.

An investigation of the effect of the relative barrier β on the accuracy of the solution of the problem was carried out for the following values: $\chi_1 = 0.3$, $\chi_2 = 3$, $\chi_3 = 30$. The results of simulation are given in Fig. 3b.

As simulation has shown, the inclusion of a barrier mechanism in the general case does not reduce the calculation time for solving the problem with specified accuracy attainable for a particular value of a barrier. However, in a number of problems of conjugated and complicated heat exchange in which there are variable boundary conditions and, consequently, in which there is no necessity to look for the exact solution of a nonlinear system, the use of a barrier results in an improvement in the calculating time on each iteration for a nonlinear system.

Conclusions. A numerical method is offered for solving the system of nonlinear equations arising from the algebraic approximation of equations of radiation and convection energy transport. The classic Newton–Raphson scheme is proposed as a foundation of the method. The use of the method of conjugate gradients with preliminary conditioning and symmetrization of the matrix of the linearized system results in considerable improvement in the computer memory and computational time as compared with the Newton–Raphson scheme in systems with the number of variables exceeding 100.

An application of preconditioning on the basis of the substitution of variables (12) allows one to solve effectively the problems of complicated heat exchange in systems with pairs of "strongly interacting" zones, such as thin screens.

NOTATION

T, temperature vector; S, vector of external sources in the zones; R, matrix of coefficients of radiation heat exchange; A, convective matrix, B, matrix of linearized system; σ, Stefan–Boltzmann constant; λ, coefficient of thermal conductivity; h, plate thickness or thickness of the pipe wall; α, coefficient of absorption of heat carrier; W, water equivalent of radiating gas; T'_g, temperature of heat carrier at the input of the unit.

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SOLITARY STRESS WAVES IN A NONLINEAR THERMOELASTIC MEDIUM

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UDC 539.3

The propagation process of solitary stress waves in a medium with five thermoelastic characteristics is investigated within the one-dimensional statement. Existence conditions and geometric characteristics of solitary waves are obtained, and restrictions are found for the elastic and thermal constants.

1. Statement of the Problem. Propagation of one-dimensional waves in a thermoelastic medium in the absence of heat sources and sinks is described within five-constant nonlinear thermoelastic theory by the system of equations [1-3]:

$$\varepsilon = e + \frac{1}{2} e^2, \tag{1}$$

$$\sigma = c_1 e + c_2 e^2 - T, \tag{2}$$

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